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Covariant path integral for the Dirac equation with pseudoscalar potentials

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Abstract
The problem of a relativistic spinning particle interacting with pseudoscalar potentials in (3 + 1) dimensions is formulated in the framework of covariant supersymmetric path integrals. The relative Green's function is expressed through a functional integral over bosonic trajectories that describe the external motion and fermionic variables that describe the spin degrees of freedom. As an application, we have considered the case of the plane wave, where the pseudoscalar potential is an arbitrary function of the variable $(k_\mu x^\mu)$. The (3 + 1)-dimensional problem is reduced to a (1 + 1)-dimensional one by using an identity. For the case of $k^2 = 0$, the relative propagator is exactly calculated and the wavefunctions are extracted.

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1. Introduction

As we know, Feynman introduced his famous path integral quantization method in order to satisfy the need for comprehension of quantum mechanics [1]. Many problems in nonrelativistic quantum mechanics are exactly solved by the use of path integral approach starting from their classical origins (i.e. classical actions). Furthermore, the path integral remains a useful quantization procedure mainly when it becomes, like in cosmology, difficult to use the other methods [2], which explains the increased interest of this method and the importance to develop the path integration techniques [3, 4].

In relativistic quantum mechanics and particularly for the Dirac equation the Feynman method has not had the same development because of the difficulty of inserting the anticommuting γ-matrices by means of paths and the fact that the spin has no classical origin. However, a successful supersymmetric formulation for relativistic spinning particles was elaborated by Fradkin and Gitman [5] according to the Feynman standard form

$$\sum_{\text{paths}} \exp \mathrm{i} S(\text{path}), \tag{1}$$

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where the supersymmetric action S describes at the same time the external motion and the internal one related to the spin of the particle. Elsewhere, the same problem is reconsidered following the so-called global and local representations by Alexandrou *et al* [6]. We note, also, that the Fradkin–Gitman formulation is generalized to the case of arbitrary dimensions in [7] and to the case of the Dirac equation with torsion field in [8].

On the other hand, the problem of the Dirac equation with pseudoscalar potentials (PSP) has been widely discussed [9–13]. The subject of spin $\frac{1}{2}$ fermion interacting with PSP has more significant implications in quantum field theory. Namely, the Dirac equation with PSP that is written in the form

$$\left[\mathrm{i}\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - \gamma^{5}V_{p}(x) - m\right]\psi(x) = 0, \tag{2}$$

with the convention $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$, is the first approximation of the field theory describing the interaction of the fermion with a pseudoscalar particle. This interaction is governed by the Lagrangian

$$\mathcal{L}_{\mathrm{int}} = \mathrm{i}g\bar{\psi}(x)\gamma^5\varphi(x)\psi(x) \tag{3}$$

where g is a coupling constant, $\psi(x)$ is the fermion field and $\varphi(x)$, which is the field of the pseudoscalar particle, is a solution of the Klein–Gordon equation in contrast to the electromagnetic interaction where the photon field A_μ obeys Maxwell's equations.

During the last few years, some considerable investigations have been made to understand the quantum behaviour of a Dirac particle subjected to PSP; De-Castro has discussed, in $(1 + 1)$ dimension, the existence of bounded states [10] in comparison with the case of $(3 + 1)$ dimension where there are no bounded solutions [11]. He also studied the problem of scattering of fermions by a pseudoscalar potential, more particularly with a step barrier and the novelty in this shape of interactions is the absence of the Klein's paradox [12]. Moreover, the pseudoscalar interactions are analysed in the context of the $(1 + 1)$-dimensional Dirac equation with non-Hermitian interactions but real energies. It is shown that the relevant hidden symmetry of the Dirac equation with such an interaction is pseudo-supersymmetry [13].

Recently, we have proposed a straightforward method for solving the problem of a Dirac particle subjected to a pseudoscalar potential in $(1+1)$ dimension by the use of supersymmetric path integrals [14]. This method proved most fruitful in finding analytical and exact expressions of the wavefunctions and the energy spectrum of the fermion.

In the present paper, which can be regarded as an extension of the previous one [14], we suggest performing a path integration for the Dirac equation with pseudoscalar potential in the more realistic $(3 + 1)$-dimensional world. In the first stage, we generalize the path integral formulation given in the previous work following the global projection. Next, we consider the plane wave case where the pseudoscalar potential is an arbitrary function of the variable $(k \cdot x)$. In the second stage, we show, after integrating over odd trajectories, that the relative Green's function can be expressed only through bosonic path integrals. Then by incorporating an identity, the $(3 + 1)$-dimensional problem will be reduced to a $(1 + 1)$-dimensional problem. For the case of $k^2 \neq 0$, the calculation will be reduced to the propagator of a Schrödinger particle in an effective supersymmetric potential. For the case of $k^2 = 0$, we show that the integration over bosonic trajectories is straightforward. Then we can easily extract the wavefunctions.

2. The general method

As is shown in [14], for the case of the pseudoscalar interaction, it is difficult to build a local path integral representation using bosonic proper time (Schwinger parameter) and fermionic one (Grassmannian variable). So, we construct a global representation starting from the causal Green's function $S^c(x_b, x_a)$ solution of the equation

$$[\gamma^\mu P_\mu - \gamma^5 V_p(x) - m] S^c(x_b, x_a) = -\delta^4(x_b - x_a). \tag{4}$$

It is known that $S^c(x_b, x_a)$ can be presented as a matrix element of an operator $\mathbb{S}^c$

$$S^c_{ij}(x_b, x_a) = \langle x_b | \mathbb{S}^c_{ij} | x_a \rangle, \tag{5}$$

with

$$[(\gamma^\mu)_{ij} P_\mu - (\gamma^5)_{ij} V_p(x) - m\delta_{ij}] \mathbb{S}^c_{jk} = -\delta_{ik}. \tag{6}$$

Here, the spinor indices i, j and k (with a sum over j) are written explicitly for clarity and will be omitted hereafter.

The operator $\mathbb{S}^c$ can be presented as follows:

$$\mathbb{S}^c = \frac{-1}{K_-} = -K_+ \frac{1}{K_- K_+} \tag{7}$$

where the operators K_- and K_+ are given by

$$K_\pm = [\gamma^\mu P_\mu - \gamma^5 V_p(x) \pm m]. \tag{8}$$

Note that this procedure is used in [6] to derive path integral representations for the propagator systematically without the usual five-dimensional extension (i.e. without γ^5) and it is also employed in [7] in the case of odd dimensions where there is no γ^5 matrix. However, in the present case, although γ^5 exists, we must use this procedure to obtain a Bose-type operator that has a quadratic form with respect to γ-matrices.

Taking into account that $[\gamma^m, \gamma^n]_+ = \eta^{mn}$, with $m, n = \overline{0, 3}, 5$ and $\eta^{mn} = \mathrm{diag}(1, -1, -1, -1, -1)$, the product $K_- K_+$ can be rearranged as follows:

$$K_- K_+ = P^2 - m^2 - V_p^2(x) + \mathrm{i}\tfrac{1}{2} F_{mn} \gamma^m \gamma^n, \tag{9}$$

where the antisymmetric tensor F_{mn}, that has to be understood as a matrix with lines marked by the first contravariant indices and with columns marked by the second covariant indices, is given by

$$\begin{aligned} F_{5\mu} &= -F_{\mu 5} = \frac{\partial}{\partial x^\mu} V_p(x), \\ F_{\mu\nu} &= 0. \end{aligned} \tag{10}$$

Now, in order to construct a global representation we use the relation $\int \mathrm{d}x |x\rangle\langle x| = 1$. We get

$$S^c(x_b, x_a) = \left[\mathrm{i}\gamma^\mu \frac{\partial}{\partial x_b^\mu} - \gamma^5 V_p(x_b) + m\right] G^c(x_b, x_a), \tag{11}$$

where the Green's function $G^c(x_b, x_a)$, that we suggest to calculate via path integration, has the following proper time representation:

$$G^c(x_b, x_a) = \mathrm{i} \int \mathrm{d}\lambda \langle x_b | \exp(-\mathrm{i}\mathcal{H}(\lambda)) | x_a \rangle, \tag{12}$$

with

$$\mathcal{H}(\lambda) = \lambda\left(-P^2 + m^2 + V_p^2(x) - \frac{\mathrm{i}}{2} F_{mn} \gamma^m \gamma^n\right). \tag{13}$$

The operator $[\mathrm{i}\gamma^\mu \partial_\mu - \gamma^5 V_p(x) + m]$ will eliminate the superfluous states caused by the product $K_- K_+$ in (7).

To present $G^c(x_b, x_a)$ by means of path integrals we write, to begin with, $\exp(-\mathrm{i}\mathcal{H}(\lambda)) = [\exp(-\mathrm{i}\mathcal{H}(\lambda)\varepsilon)]^N$, with $\varepsilon = 1/N$, and we insert $(N - 1)$ identities $\int \mathrm{d}x |x\rangle\langle x| = 1$ between all the operators $\exp(-\mathrm{i}\varepsilon\mathcal{H}(\lambda))$. Next, we introduce N integrations $\int \mathrm{d}\lambda_k \delta(\lambda_k - \lambda_{k-1}) = 1$. We obtain

$$G^c = \mathrm{i} \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \int \mathrm{d}\lambda_0 \int \mathrm{d}x_1\, \mathrm{d}x_2 \cdots \mathrm{d}x_{N-1} \int \mathrm{d}\lambda_1\, \mathrm{d}\lambda_2 \cdots \mathrm{d}\lambda_N \times \prod_{k=1}^{N} \langle x_k | \exp(-\mathrm{i}\varepsilon\mathcal{H}(\lambda_k)) | x_{k-1}\rangle \delta(\lambda_k - \lambda_{k-1}). \tag{14}$$

Now, we must express the matrix elements $\langle x_k | \exp(-\mathrm{i}\varepsilon\mathcal{H}(\lambda_k)) | x_{k-1}\rangle$ through path integral. As ε is small, we can write

$$\langle x_k | \exp(-\mathrm{i}\varepsilon\mathcal{H}(\lambda_k)) | x_{k-1}\rangle \approx \langle x_k | 1 - \mathrm{i}\varepsilon\mathcal{H}(\lambda_k) | x_{k-1}\rangle. \tag{15}$$

Then, we insert into (15) the integral identity $\int \mathrm{d}p_k |p_k\rangle\langle p_k| = 1$. By taking into account that $\mathcal{H}(\lambda)$ has no product of the operators X, P and by using the relation

$$\langle x_k \mid p_{k'}\rangle = \frac{1}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}p_{k'}x_k}, \tag{16}$$

the matrix element (15) can be expressed in the middle point $\tilde{x}_k = (x_k + x_{k-1})/2$

$$\langle x_k | \exp(-\mathrm{i}\varepsilon\mathcal{H}(\lambda_k)) | x_{k-1}\rangle = \int \frac{\mathrm{d}p_k}{(2\pi)^2} \exp\left\{\mathrm{i}\left[p_k \frac{x_k - x_{k-1}}{\varepsilon} - \mathcal{H}(\lambda_k, \tilde{x}_k, p_k)\right]\varepsilon\right\}. \tag{17}$$

Since the multipliers in (14) are noncommutative due to the γ-matrices structure, we attribute formally the index k, to γ-matrices, and we introduce the $\mathbb{T}$-product which acts on γ-matrices. Then, using the integral representation for the δ-functions

$$\delta(\lambda_k - \lambda_{k-1}) = \frac{\mathrm{i}}{2\pi} \int \mathrm{e}^{\mathrm{i}\pi_k(\lambda_k - \lambda_{k-1})}\, \mathrm{d}\pi_k, \tag{18}$$

it becomes possible to gather all the multipliers, entering in (14), in one exponent. The Green's function G^c will be then expressed as follows:

$$G^c = \mathbb{T} \int_0^\infty \mathrm{d}\lambda_0 \int \mathrm{D}x \int \mathrm{D}p \int \mathrm{D}\lambda \int \mathrm{D}\pi \times \exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau\left[\lambda\left(p^2 - m^2 - V_p^2(x)\right) + p\cdot\dot{x} + \pi\dot{\lambda} + \lambda\frac{\mathrm{i}}{2}F_{mn}\gamma^m\gamma^n\right]\right\}. \tag{19}$$

In order to insert the γ-matrices by means of path integrals we introduce odd sources ρ^μ. We obtain

$$G^c = \mathbb{T} \int_0^\infty \mathrm{d}\lambda_0 \int \mathrm{D}x \int \mathrm{D}p \int \mathrm{D}\lambda \int \mathrm{D}\pi \times \exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau\left[\lambda\left(p^2 - m^2 - V_p^2(x)\right) + p\cdot\dot{x} + \pi\dot{\lambda} + \lambda\frac{\mathrm{i}}{2}F_{mn}\frac{\delta_\ell}{\delta\rho^m}\frac{\delta_\ell}{\delta\rho^n}\right]\right\} \mathbb{T}\exp\int_0^1 \rho(\tau)\gamma\, \mathrm{d}\tau\Big|_{\rho=0}. \tag{20}$$

Next, we present the quantity $\mathbb{T}\exp\int_0^1 \rho(\tau)\gamma\,\mathrm{d}\tau$ via a path integral over Grassmannian trajectories [5, 6]

$$\mathbb{T}\exp\int_0^1 \rho(\tau)\gamma\,\mathrm{d}\tau = \exp\left(\mathrm{i}\gamma^n\frac{\partial_l}{\partial\theta^n}\right)\int_{\psi(0)+\psi(1)=\theta}\mathcal{D}\psi \times \exp\left\{\int_0^1 \mathrm{d}\tau[\psi_n\dot{\psi}^n - 2\mathrm{i}\rho_n\psi^n] + \psi_n(1)\psi^n(0)\right\}, \tag{21}$$

where the measure $\mathcal{D}\psi$ is given by

$$\mathcal{D}\psi = D\psi \left[\int_{\psi(0)+\psi(1)=0} D\psi \exp\left\{\int_0^1 \psi_n \dot{\psi}^n \,\mathrm{d}\tau\right\}\right]^{-1} \tag{22}$$

and θ^n and ψ^n are odd variables, anticommuting with γ-matrices.

Finally, the Green's function G^c takes the following Hamiltonian path integral representation:

$$G^c = \exp\left(\mathrm{i}\gamma^n \frac{\partial_l}{\partial\theta^n}\right) \int_0^\infty \mathrm{d}\lambda_0 \int \mathrm{D}x \int \mathrm{D}p \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi$$

$$\int D\lambda \int D\pi \exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau\left[\lambda\left(p^2 - m^2 - V_p^2(x) + 2\mathrm{i}F_{mn}\psi^m\psi^n\right)\right.\right.$$

$$\left.\left.\left. - \mathrm{i}\psi_n\dot{\psi}^n + p\cdot\dot{x} + \pi\dot{\lambda}\right] + \psi_n(1)\psi^n(0)\right\}\right|_{\theta=0}. \tag{23}$$

We note that integrating over momenta and separating the gauge-fixing term $\pi\dot{\lambda}$ and the boundary term $\psi_n(1)\psi^n(0)$ we obtain the super-gauge invariant action

$$\mathcal{A} = \int_0^1 \left[-\frac{\dot{x}^2}{4\lambda} - \lambda V_p^2(x) - \mathrm{i}\psi_n\dot{\psi}^n + 2\mathrm{i}\lambda F_{mn}\psi^m\psi^n\right]\mathrm{d}\tau, \tag{24}$$

which resembles the Berezin–Marinov action [15–18]. From this action we can easily deduce the Lagrangian classical equations of motion

$$\frac{\ddot{x}_\mu}{2\lambda} - 2\lambda V_p(x)\frac{\partial}{\partial x^\mu}V_p(x) + 2\lambda\mathrm{i}\left(\frac{\partial}{\partial x^\mu}F_{mn}\right)\psi^m\psi^n = 0, \tag{25}$$

$$-2\mathrm{i}\dot{\psi}_m + 4\mathrm{i}\lambda F_{mn}\psi^n = 0. \tag{26}$$

Having shown how to formulate the problem of Dirac particle interacting with a pseudoscalar potential in the framework of Feynman–Beresin path integral, let us examine our method by elaborating an explicit example.

3. Application

In order to examine this method let us choose the pseudoscalar plane wave potential that is given by

$$V_p(x) = gf(k\cdot x), \tag{27}$$

where $f(\phi)$ is an arbitrary function of the variable $\phi = k\cdot x$ and k_μ is the propagation vector. In this case, the antisymmetric tensor F_{mn} will be

$$F_{mn} = gf'(k\cdot x)\mathfrak{f}_{nm}, \tag{28}$$

with

$$\mathfrak{f}_{5\mu} = -\mathfrak{f}_{\mu 5} = k_\mu, \qquad \mathfrak{f}_{\mu\nu} = 0. \tag{29}$$

To begin, let us fix in equation (23) the gauge over the proper time λ by performing the functional integral over π and λ. The Green's function will take the following form:

$$G^c = \exp\left(\mathrm{i}\gamma^n \frac{\partial_l}{\partial\theta^n}\right) \int_0^\infty \mathrm{d}\lambda \int \mathrm{D}x \int \mathrm{D}p \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi$$

$$\left.\times \exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau\left[\lambda\left(p^2 - m^2 - V_p^2(x)\right) + p\cdot\dot{x}\right]\right\}\mathcal{I}(x,\lambda,\theta)\right|_{\theta=0}, \tag{30}$$

where the factor $\mathcal{I}(x, \lambda, \theta)$ is given by

$$\mathcal{I}(x, \lambda, \theta) = \int_{\psi(0)+\psi(1)=\theta} \mathcal{D}\psi \exp\left\{\int_0^1 \mathrm{d}\tau[\psi_n\dot{\psi}^n - 2\lambda F_{mn}\psi^m\psi^n] + \psi_n(1)\psi^n(0)\right\}. \tag{31}$$

Let us now integrate over Grassmannian variables, to express G^c only through bosonic path integrals. Since the integration variables ψ obey the boundary condition $\psi(0) + \psi(1) = \theta$, it is suitable, in order to calculate $\mathcal{I}(x, \lambda, \theta)$, to change ψ by ξ, where

$$\psi = \frac{1}{2}\xi + \frac{\theta}{2}. \tag{32}$$

The new variables ξ obey the following boundary condition:

$$\xi(0) + \xi(1) = 0. \tag{33}$$

Then, in order to obtain a more familiar form with respect to ξ variables, we change the proper time from τ to σ, where

$$\mathrm{d}\sigma = f'(k\cdot x)\,\mathrm{d}\tau. \tag{34}$$

The factor $\mathcal{I}(x, \lambda, \theta)$ will be given through the Grassmann Gaussian integral

$$\mathcal{I}(x, \lambda, \theta) = \exp\left(-\frac{g}{2}\tilde{\lambda}\mathfrak{f}_{nm}\theta^n\theta^m\right)\int \mathcal{D}\xi$$
$$\times \exp\left\{\int_0^1\left[\frac{1}{4}\xi_n\dot{\xi}^n - \frac{1}{2}g\tilde{\lambda}\mathfrak{f}_{nm}\xi^n\xi^m - g\tilde{\lambda}\mathfrak{f}_{nm}\theta^n\xi^m\right]\mathrm{d}\sigma\right\}\Big|_{\theta=0}, \tag{35}$$

where

$$\tilde{\lambda} = \lambda\int_0^1 f'(k\cdot x)\,\mathrm{d}\tau. \tag{36}$$

Since $\mathfrak{f}_{nm}$ is constant, the problem is reduced to the constant electromagnetic field with the five-dimensional extension.

$\mathcal{I}(x, \lambda, \theta)$ can then be evaluated to be

$$\mathcal{I}(x, \lambda, \theta) = \det{}^{\frac{1}{2}}\left[\frac{M(g)}{M(g=0)}\right]\exp\left(-\frac{g}{2}\tilde{\lambda}\mathfrak{f}_{nm}\theta^n\theta^m\right)$$
$$\times \exp\left\{\int_0^1[\mathcal{J}^m(\sigma')(M^{-1})_{mn}\mathcal{J}^n(\sigma)]\,\mathrm{d}\sigma'\,\mathrm{d}\sigma\right\}, \tag{37}$$

where the matrix M and the current $\mathcal{J}$ are given by

$$M_{mn}(g) = \eta_{mn}\delta'(\sigma-\sigma') - 2g\mathfrak{f}_{mn}\delta(\sigma-\sigma'), \tag{38}$$

and

$$\mathcal{J}_n = g\tilde{\lambda}\mathfrak{f}_{nm}\theta^m. \tag{39}$$

The determinant in (37) can be written as

$$\det\left[\frac{M(g)}{M(0)}\right] = \exp\{\mathrm{Tr}[\log M(g) - \log M(0)]\}$$
$$= \exp\left\{-\mathrm{Tr}\int_0^g \mathrm{d}g'\int \mathrm{d}\sigma\int \mathrm{d}\sigma'\mathcal{R}(g';\sigma,\sigma')\mathfrak{f}\right\}, \tag{40}$$

where the tensor $\mathcal{R}_{mn}(g;\sigma,\sigma')$ is a solution of the equation (see [19])

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathcal{R}_{mn}(g;\sigma,\sigma') - g\mathfrak{f}_m{}^l\mathcal{R}_{ln}(g;\sigma,\sigma') = \eta_{mn}\delta(\sigma-\sigma'), \tag{41}$$

with the boundary conditions

$$\mathcal{R}_{mn}(g; 1, \sigma) = -\mathcal{R}_{mn}(g; 0, \sigma), \qquad \forall\sigma \in [0, 1]. \tag{42}$$

Writing $\mathcal{R}$ in its explicit form

$$\mathcal{R} = \left(\tfrac{1}{2}\eta\varepsilon(\sigma - \sigma') - \tfrac{1}{2}\tanh(g\tilde{\lambda}\mathfrak{f})\right)\exp[g\tilde{\lambda}\mathfrak{f}(\sigma - \sigma')], \tag{43}$$

and using the property

$$\exp[\mathrm{Tr}(\ln A)] = \det(A), \tag{44}$$

we obtain

$$\det\left[\frac{M(g)}{M(0)}\right] = \det(\cosh g\tilde{\lambda}\mathfrak{f}). \tag{45}$$

(Note that $\varepsilon(\sigma - \sigma')$ is the sign of $(\sigma - \sigma')$.)

Then, the factor $\mathcal{I}(x, \lambda, \theta)$ will be rearranged as follows:

$$\mathcal{I}(x, \lambda, \theta) = \det{}^{\frac{1}{2}}(\cosh g\tilde{\lambda}\mathfrak{f})(1 - B_{nm}\theta^n\theta^m), \tag{46}$$

where the tensor B_{nm} that has to be understood as a matrix is given by

$$B = \tfrac{1}{2}\tanh(g\tilde{\lambda}\mathfrak{f}). \tag{47}$$

At this level we distinguish the following two different cases.

Case 1: $k^2 \neq 0$

In this case, being aware of $\mathfrak{f}^3 = k^2\mathfrak{f}$, we easily obtain

$$\mathcal{I}(x, \lambda, \theta) = \cos(g|k|\tilde{\lambda}) + \frac{1}{2}\sin(g|k|\tilde{\lambda})\frac{k_\mu}{|k|}(\theta^\mu\theta^5 - \theta^5\theta^\mu). \tag{48}$$

Then, the spin factor will take the form

$$\exp\left(\mathrm{i}\gamma^n\frac{\partial_l}{\partial\theta^n}\right)\mathcal{I}(x, \lambda, \theta)\bigg|_{\theta=0} = \sum_{s=\pm1}\frac{1}{2}\left(1 + s\frac{\mathrm{i}\hat{k}\gamma^5}{|k|}\right)\exp\left(\mathrm{i}\lambda s|k|\int_0^1 V_p'(k\cdot x)\,\mathrm{d}\tau\right), \tag{49}$$

where we have used the notation $\hat{k} = k = \gamma^\mu k_\mu$ and $|k| = \sqrt{-k_0^2 + \vec{k}^2} = \sqrt{-k^\mu k_\mu}$.

The Green's function G^c will be given by

$$G^c = \sum_{s=\pm1}\frac{1}{2}\left(1 + s\frac{\mathrm{i}\hat{k}\gamma^5}{|k|}\right)\int_0^\infty \mathrm{d}\lambda\int \mathrm{D}x\int \mathrm{D}p\exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau\left[\lambda\left(p^2 - m^2 - V_p^2(k\cdot x)\right)\right.\right.$$
$$\left.\left. + s\lambda|k|V_p'(k\cdot x) + p\cdot\dot{x}\right]\right\}\bigg|_{\theta=0}. \tag{50}$$

In order to continue we benefit from the feature of the plane wave by incorporating the identity [20]

$$\int \mathrm{d}\phi_a\,\mathrm{d}\phi_b\,\delta(\phi_a - k\cdot x_a)\int \mathrm{D}\phi\,\mathrm{D}p_\phi\exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau\; p_\phi(\dot{\phi} - k\dot{x})\right\} = 1, \tag{51}$$

which makes the variable $k\cdot x$ independent of the quadri-vector of position x. Then, by making the shift $p - p_\phi k \to p$, it becomes possible to integrate over x and p. The integration over x

gives $\delta(\dot{p})$, that means that p is constant. We get

$$
\begin{aligned}
G^c = \sum_{s=\pm 1} \frac{1}{2}\left(1 + s\frac{\mathrm{i}\hat{k}\gamma^5}{|k|}\right) \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p(x_b - x_a)} \\
\times \int \mathrm{d}\phi_a\, \mathrm{d}\phi_b\, \delta(\phi_a - k\cdot x_a) \int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{\mathrm{i}\lambda(p^2 - m^2)} \\
\times \int \mathrm{D}\phi\, \mathrm{D}p_\phi \exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau\big[\lambda\big(k^2 p_\phi^2 - V_p^2(\phi)\big)\right. \\
\left. \times\, \lambda s|k|V_p'(\phi) + p_\phi(\dot{\phi} + 2\lambda p k)\big]\right\}.
\end{aligned}
\tag{52}
$$

Integrating over p_ϕ, we obtain a simple expression for the Green's function G^c

$$
\begin{aligned}
G^c = \sum_{s=\pm 1} \frac{1}{2}\left(1 + s\frac{\mathrm{i}\hat{k}\gamma^5}{|k|}\right) \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p(x_b - x_a)} \int_0^{+\infty} \mathrm{d}\lambda \exp\left[\mathrm{i}\lambda\left(p^2 - m^2 - \frac{(p\cdot k)^2}{k^2}\right)\right] \\
\times \int \mathrm{d}\phi_a\, \mathrm{d}\phi_b\, \delta(\phi_a - k\cdot x_a) K_s(\phi_b, \phi_a, \lambda),
\end{aligned}
\tag{53}
$$

where

$$
\begin{aligned}
K_s(\phi_b, \phi_a, \lambda) = \exp\left\{-\mathrm{i}\frac{p\cdot k}{k^2}(\phi_b - \phi_a)\right\} \\
\times \int \mathrm{D}\phi \exp\left\{\mathrm{i}\int_0^\lambda \mathrm{d}\tau\left[\left(\frac{\dot{\phi}^2}{4|k|^2} - V_p^2(\phi) + s|k|V_p'(\phi)\right)\right]\right\}.
\end{aligned}
\tag{54}
$$

Thus, the calculation is reduced to the propagator of Schrödinger particle in the effective supersymmetric potential $U(\phi) = V_p^2(\phi) - s|k|V_p'(\phi)$. The $(3+1)$-dimensional problem is reduced simply to a one-dimensional propagator. In other words, the motion of the fermion in $(3+1)$ dimensions is projected along direction of the wave vector.

Also, if we take $k_\mu = (0, -1, 0, 0)$ we make contact with the one-dimensional problem studied in the previous work ($V_p(x) = V_p(x^1)$). By performing a simple integration over ϕ_a in (53) followed by integration over p^1 and ϕ_b one can find

$$
\begin{aligned}
G^c = \sum_{s=\pm 1} \frac{1}{2}\left(1 + s\frac{\mathrm{i}\hat{k}\gamma^5}{|k|}\right) \times \int \frac{\mathrm{d}p^0}{2\pi}\frac{\mathrm{d}^2 p_\perp}{(2\pi)^2} \\
\times \exp\big(\mathrm{i}\big[p^0\big(x_b^0 - x_a^0\big) - p_\perp(x_{\perp b} - x_{\perp a})\big]\big) P_s\big(x_b^1, x_a^1\big),
\end{aligned}
\tag{55}
$$

where $p_\perp = (p^2, p^3) \equiv (p_y, p_z)$, $x_\perp = (x^2, x^3) \equiv (y, z)$ and the kernel $P_s\big(x_b^1, x_a^1\big)$ is the propagator of a nonrelativistic particle subjected to a supersymmetric potential:

$$
\begin{aligned}
P_s\big(x_b^1, x_a^1\big) = \int_0^{+\infty} \mathrm{d}\lambda \exp\big[\mathrm{i}\lambda\big(p_0^2 - p_\perp^2 - m^2\big)\big] \\
\times \int \mathrm{D}x^1 \exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau\left[\left(\frac{(\dot{x}^1)^2}{4\lambda} - \lambda V_p^2(x^1) + s\lambda V_p'(x^1)\right)\right]\right\}.
\end{aligned}
\tag{56}
$$

Case 2: $k^2 = 0$

In this case, we have $\mathfrak{f}^3 = 0$. So, it is easy to show that

$$B = \tfrac{1}{2}\tilde{\lambda}\mathfrak{f} \tag{57}$$

and

$$\mathcal{I}(x, \lambda, \theta) = \left(1 + \frac{\tilde{\lambda}}{2} k_\mu (\theta^\mu \theta^5 - \theta^5 \theta^\mu)\right). \tag{58}$$

Then, the spin factor will take the form

$$\exp\left(\mathrm{i}\gamma^n \frac{\partial_l}{\partial \theta^n}\right) \mathcal{I}(x, \lambda, \theta)\Big|_{\theta=0} = 1 - \hat{k}\gamma^5 \lambda \int_0^1 V_p'(k \cdot x)\, \mathrm{d}\tau \tag{59}$$

and, consequently, the Greens function G^c can be expressed only through bosonic path integrals

$$G^c = \int_0^\infty \mathrm{d}\lambda \int \mathrm{D}p \int \mathrm{D}x \left[1 - \hat{k}\gamma^5 \lambda \int_0^1 V_p'(k \cdot x)\, \mathrm{d}\tau\right] \times \exp\left\{\mathrm{i} \int_0^1 \mathrm{d}\tau\, \lambda \left(p^2 - m^2 - V_p^2(x)\right) + p\dot{x}\right\}. \tag{60}$$

As previously by incorporating the identity (51) and by making the shift $p - p_\phi k \to p$, we get

$$G^c = \int \mathrm{d}\phi_a\, \mathrm{d}\phi_b\, \delta\left(\phi_a - k \cdot x_a\right) \int_0^\infty \mathrm{d}\lambda \times \int \mathrm{D}\phi\, \mathrm{D}p_\phi \int \mathrm{D}x\, \mathrm{D}p \left(1 - \hat{k}\gamma^5 \lambda \int_0^1 V_p'(\phi)\, \mathrm{d}\tau\right) \times \exp\left\{\mathrm{i} \int_0^1 \left[\lambda\left(p^2 - m^2 - V_p^2(\phi)\right) + p_\phi(\dot{\phi} + 2\lambda pk) + p\dot{x}\right] \mathrm{d}\tau\right\}. \tag{61}$$

The integration over p_ϕ gives a delta functional $\delta(\dot{\phi} + 2pk)$ that is related directly by the Lagrangian equation of motion projected in the direction of the plane wave. By vanishing the argument of $\delta(\dot{\phi} + 2pk)$, we get

$$\mathrm{d}\tau = -\frac{\mathrm{d}\phi}{2\lambda pk}. \tag{62}$$

Integrating now over the plane wave variable ϕ we obtain the final expression of the Green's function G^c

$$G^c = \int_0^\infty \mathrm{d}\lambda \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p\cdot(x_b - x_a)}\, \mathrm{e}^{\mathrm{i}\lambda\left(p^2 - m^2\right)} \times \left[1 + \frac{\hat{k}\gamma^5}{2pk}(V_p(k \cdot x_b) - V_p(k \cdot x_a))\right] \exp\left\{\mathrm{i}\frac{1}{2pk} \int_{k\cdot x_a}^{k\cdot x_b} V_p^2(\phi)\, \mathrm{d}\phi\right\}. \tag{63}$$

In order to symmetrize this expression we write

$$\left(1 + \frac{\hat{k}\gamma^5}{2pk}(b - a)\right) = \left(1 - \frac{\hat{k}\gamma^5}{2pk} a\right)\left(1 + \frac{\hat{k}\gamma^5}{2pk} b\right) \tag{64}$$

and we do integration over λ

$$G^c = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}p\cdot(x_b - x_a)} \left(1 + \frac{\hat{k}\gamma^5}{2pk} V_p\left(k \cdot x_b\right)\right) \times \frac{1}{p^2 - m^2 + \mathrm{i}\epsilon} \exp\left\{\mathrm{i}\frac{1}{2pk} \int_{k\cdot x_a}^{k\cdot x_b} V_p^2(\phi)\, \mathrm{d}\phi\right\} \left(1 - \frac{\hat{k}\gamma^5}{2pk} V_p(k \cdot x_a)\right). \tag{65}$$

Now, we change p by $-p$ in the last expression of G^c and we incorporate it in equation (11) to obtain the closed expression of $S^c(x_b, x_a)$:

$$S^c(x_b, x_a) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \mathrm{e}^{-\mathrm{i}p\cdot(x_b - x_a)} \left(1 + \frac{\hat{k}\gamma^5}{2pk} V\,(k\cdot x_b)\right)$$
$$\times \frac{\hat{p} + m}{p^2 - m^2 + \mathrm{i}\epsilon} \exp\left\{-\mathrm{i}\frac{1}{2pk}\int_{k\cdot x_a}^{k\cdot x_b} V_p^2(\phi)\,\mathrm{d}\phi\right\}\left(1 - \frac{\hat{k}\gamma^5}{2pk} V(k\cdot x_a)\right). \qquad (66)$$

In order to determine the wavefunctions, let us integrate over the energy p^0 and employ the projectors of the positive and negative energy states [21]:

$$\Lambda_+(p) = \sum_{\pm s} u\,(p, s)\,\bar{u}\,(p, s) = \frac{\hat{p} + m}{2m}, \qquad (67)$$

$$\Lambda_-(p) = -\sum_{\pm s} v(p, s)\bar{v}(p, s) = \frac{-\hat{p} + m}{2m}. \qquad (68)$$

We then obtain for $S^c(x_b, x_a)$ the following form:

$$S^c(x_b, x_a) = -\mathrm{i}\theta\,(t_b - t_a) \int \mathrm{d}^3 p \sum_{\pm s} \psi^{(+)}_{s,\mathbf{p}}(x_b)\bar{\psi}^{(+)}_{s,\mathbf{p}}(x_a)$$
$$+\mathrm{i}\theta\,(t_a - t_b) \int \mathrm{d}^3 p \sum_{\pm s} \psi^{(-)}_{s,\mathbf{p}}(x_b)\bar{\psi}^{(-)}_{s,\mathbf{p}}(x_a), \qquad (69)$$

where

$$p^0 = (\vec{p}^2 + m^2)^{1/2}, \qquad (70)$$

and the wavefunctions are given by

$$\psi^{(+)}_{s,\mathbf{p}}(x) = \frac{1}{(2\pi)^{3/2}}\left(\frac{m}{p^0}\right)^{1/2}\left[1 + \frac{\hat{k}\gamma^5}{2pk} V_p(x)\right]$$
$$\times u(p, s) \times \exp\left\{-\mathrm{i}p\cdot x - \frac{\mathrm{i}}{2pk}\int_0^{k\cdot x} V_p^2(\phi)\,\mathrm{d}\phi\right\}, \qquad (71)$$

and

$$\psi^{(-)}_{s,\mathbf{p}}(x) = \frac{1}{(2\pi)^{3/2}}\left(\frac{m}{p^0}\right)^{1/2}\left[1 - \frac{\hat{k}\gamma^5}{2pk} V_p(x)\right]$$
$$\times v\,(p, s) \times \exp\left\{\mathrm{i}p\cdot x - \frac{\mathrm{i}}{2pk}\int_0^{k\cdot x} V_p^2(\phi)\,\mathrm{d}\phi\right\}. \qquad (72)$$

Let us note at the end of this work that due to the pseudoscalar interaction that carries γ^5 one cannot derive path integral representation for the corresponding propagator without the five-dimensional extension. In addition, although some techniques used in this paper with the five-dimensional extension are the generalization of those used in the usual four-dimensional case, the obtained results are new.

4. Conclusion

In this paper, we have given a covariant path integral method to analyse the problem of a Dirac particle subjected to pseudoscalar potential in (3 + 1) dimension. The relative Green's function is presented by means of supersymmetric path integrals in the so-called global projection, where the internal motion relative to the spin of the fermion is described by odd Grassmannian variables. As an application, we have considered the case of the plane wave, where the pseudoscalar potential is an arbitrary function of the variable $(k_\mu x^\mu)$. Since the pseudoclassical action has a more familiar form with respect to ψ-variables, we were able to express the Green's function only through bosonic path integrals. Then, the (3+1)-dimensional problem is reduced to (1 + 1)-dimensional one by using the identity (51). For the case of $k^2 \neq 0$, the calculation will be reduced to the propagator of Schrödinger particle in an effective supersymmetric potential. In this case, we can easily connect to the one-dimensional problem studied previously. For the case of $k^2 = 0$, we have exactly calculated the relative propagator and we have found the wavefunctions.

Through the formulation given above and for the explicit example, we conclude, as in the previous paper, that the supersymmetric path integral is a powerful method to formulate the relativistic quantum mechanics of a Dirac particle subjected to pseudoscalar potentials.

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